2019-Sep-12 (lecture)

Thursday, September 12, 2019 1:44 AM

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MAT B44H3 - 2019

Outline:

- More substitutions
- Exact differentials
- Integrating Factors

Last Time:

We defined a separable first-order ODE

$$\dot{x}(t) = g(t) f(x) \qquad \left(\text{or, alternately } g(t)dt + f(x)dx = 0\right)$$

We can solve these by integrating $\int g(t)dt + \int f(x)dx = 0.$

We also defined homogeneous functions $f(tx, ty) = t^n f(x,y)$

And showed that an ODE $P(x,y)dx + Q(x,y)dy = 0$, with homogeneous coefficients $P(x,y)$ and $Q(x,y)$ of the same order, can be solved by substituting

$$y = ux, \quad dy = u\,d\cancel{x} + x\,du$$

which turns it into a separable ODE.

Linear Coefficients

We label ODEs of the form $(a_1 x + b_1 y + c_1)dx + (a_2 x + b_2 y + c_2)dy = 0$ as having linear coefficients

We call these linear because
$$a_1 x + b_1 y + c_1 = 0$$
$$a_2 x + b_2 y + c_2 = 0$$
define lines in the XY-plane.

What do we know how to solve? 1. separable equations

2. homogeneous coefficients

transform

We want to transform linear coefficients into

Let's start with an easy example. Suppose $c_1 = c_2 = 0$

$$(a_1 x + b_1 y)dx + (a_2 x + b_2 y)dy = 0$$

Then $(a_1 x + b_1 y)$, $(a_2 x + b_2 y)$ are already homogeneous

Let $y = ux$ $\quad dy = u\,dx + x\,du$

$$\Rightarrow (a_1 x + b_1 ux)dx + (a_2 x + b_2 ux)(u\,dx + x\,du) = 0$$

$$\Rightarrow [a_1 x + b_1 ux + a_2 ux + b_2 u^2 x]dx + [a_2 x^2 + b_2 ux^2]du = 0$$

$$\Rightarrow x(a_1 + b_1 u + a_2 u + b_2 u^2)dx + x^2(a_2 + b_2 u)du = 0$$

If $x \neq 0$, and $a_1 + b_1 u + a_2 u + b_2 u^2 \neq 0$,

$$\frac{1}{x}dx + \frac{a_2 + b_2 u}{a_1 + b_1 u + a_2 u + b_2 u^2}du = 0 \leftarrow \text{separable}$$

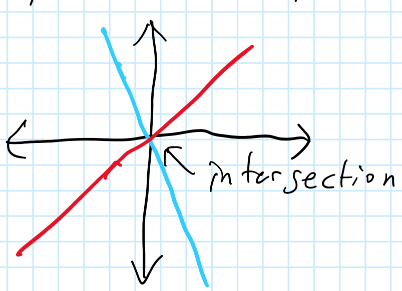
Note that you also have to check if $x = 0$
$a_1 + b_1 u + a_2 u + b_2 u^2 = 0$
are solutions

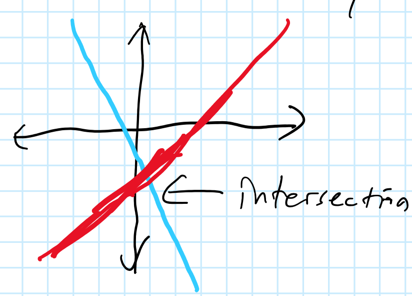
What was the key? Linear coefficients without constants are homogeneous.

Let's look at $\mathbb{R}^2$ geometry of an example.

$(2x + y)dx + (x - y)dy = 0$ $\qquad$ $(2x + y + 1)dx + (x - y - 4)dy = 0$

$\left.\begin{matrix} 2x + y = 0 \\ x - y = 0 \end{matrix}\right\} \Rightarrow \begin{cases} y = -2x \\ y = x \end{cases}$ $\qquad$ $\left.\begin{matrix} 2x + y + 1 = 0 \\ x - y - 4 = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} y = -2x - 1 \\ y = x - 4 \end{matrix}$





Having a constant term means that the lines intersect at not the origin.

Can we change the coordinate system so these lines intersect at the origin?

Method 1: Find the intersection pt and translate it

$$\left.\begin{aligned} y &= -2x-1 \\ y &= x-4 \end{aligned}\right\} \Rightarrow \begin{aligned} -2x-1 &= x-4 \\ 3x &= 3 \\ x &= 1 \\ y &= -3 \end{aligned}$$

Let $x = \bar{x} + 1 \qquad dx = d\bar{x}$

$y = \bar{y} - 3 \qquad dy = d\bar{y}$

$$(2x+y+1)\,dx + (x-y-4)\,dy = 0$$

$$(2\bar{x}+2+\bar{y}-3+1)\,d\bar{x} + (\bar{x}+1-\bar{y}+3-4)\,d\bar{y} = 0$$

homogeneous $\longrightarrow$ $(2\bar{x}+\bar{y})\,d\bar{x} + (\bar{x}-\bar{y})\,d\bar{y} = 0$

Method 2: Use the two lines as the new coordinate system

Let $u = 2x+y+1 \qquad du = 2dx + dy$

$v = x-y-4 \qquad dv = dx - dy$

$du + dv = 3dx$

$dx = \dfrac{du+dv}{3}, \quad dy = \dfrac{du-2dv}{3}$

$$u\left(\frac{du+dv}{3}\right) + v\left(\frac{du-2dv}{3}\right) = 0$$

$$u(du+dv) + v(du-2dv) = 0$$

$(u+v)\,du + (u-2v)\,dv = 0 \leftarrow$ homogeneous

This only works when the lines intersect

Method 3: If the lines don't intersect, we only need one substitution

$$(x+2y+5)\,dx + (2x+4y-3)\,dy = 0$$

$x+2y+5 = 0$

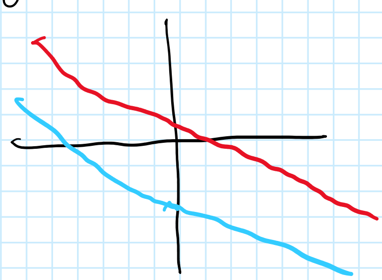
$2y = -x-5$

$y = -\dfrac{x}{2} - \dfrac{5}{2}$

$2x+4y-3 = 0$

$4y = -2x+3$

$y = -\dfrac{x}{2} + \dfrac{3}{4}$



What if we choose one line as a new variable and keep an old variable.

Let $u = x+2y+5 \qquad du = dx + 2dy$

$x = u-2y-5 \qquad dx = du - 2dy$

$u(du-2dy) + (2u-13)\,dy = 0 \leftarrow$ separable

$u\,du - 13\,dy = 0$

Exact differentials

From multivariable calculus (the coreq B41), we can define the total differential of a function $z = f(x,y)$ by

$$dz = \frac{\partial f(x,y)}{\partial x} dx + \frac{\partial f(x,y)}{\partial y} dy.$$

Ex $z = x^2 + y^2$

$dz = 2x\,dx + 2y\,dy$

Def. A differential expression

$$P(x,y)dx + Q(x,y)dy$$

is called an **exact differential** if it is the total differential of some function $f(x,y)$.

i.e. if $P(x,y) = \frac{\partial}{\partial x} f(x,y)$ and $Q(x,y) = \frac{\partial}{\partial y} f(x,y)$.

When an exact differential appears in an ODE, and if we know the function it is the total differential of, then we can easily integrate the **exact differential equation** to get a 1-parameter family of solutions

$$f(x,y) = c.$$

Ex. $(6xy + 5y)dx + (3x^2 + 5x + 3y^2)dy = 0$

$$f(x,y) = 3x^2y + 5xy + y^3$$

Then $3x^2y + 5xy + y^3 = c$ is a solution.

How do we recognize exact differentials?

Theorem 9.3 Tenenbaum

$P(x,y)dx + Q(x,y)dy = 0$ is exact if and only if

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x},$$

where $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ all exist and are continuous in a simply where $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$ all exist and are continuous in a simply connected region $R \subseteq \mathbb{R}^2$.

Ex. $2x\,dx + 3y\,dy = 0$